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Representations of fractional Brownian motion using vibrating strings ☆

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Abstract

In this paper, we show that the moving average and series representations of fractional Brownian motion can be obtained using the spectral theory of vibrating strings. The representations are shown to be consequences of general theorems valid for a large class of second-order processes with stationary increments. Specifically, we use the 1–1 relation discovered by M.G. Krein between spectral measures of continuous second-order processes with stationary increments and differential equations describing the vibrations of a string with a certain length and mass distribution.

Keywords: Fractional Brownian motion; Krein correspondence; Moving average representation; Series expansion; Vibrating string

1. Introduction

In recent years, there has been significant progress in the study of the fractional Brownian motion (fBm). In general, the analysis of this process can be rather

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complicated, since it is neither a Markov process nor a semimartingale (unless the Hurst index equals $\frac{1}{2}$). Representations of the fBm in terms of simpler, better understood processes are therefore of great importance. A key result in this respect is the moving average representation

$$X_t = \int_0^t w_t(u)\,\mathrm{d}W_u \tag{1.1}$$

of the fBm X as a stochastic integral of a deterministic kernel w_t with respect to an ordinary Brownian motion W. See for instance Molchan [19], Decreusefond and Üstünel [6], Norros et al. [21], Nuzman and Poor [23] or Pipiras and Taqqu [24] for various proofs of this result. The moving average representation allows us to obtain results regarding for instance prediction, absolute continuity, maximal inequalities, stochastic calculus, etc. (cf. e.g. [6,21,22,18,24,27], to mention but a few). More generally, it allows us to apply the results and techniques available for general Volterra processes to the fBm (see e.g. [2,3,13] for such results).

In this paper, we present an approach to (1.1) which has remained unexplored so far. The method also yields a new proof of the series representation of the fBm obtained recently in Dzhaparidze and Van Zanten [11]. The presented approach is not just yet another ad hoc method applicable only to the fBm. On the contrary, the purpose of the paper is to show that the moving average and series representations of the fBm are in fact special cases of results for a much wider class of second-order processes with stationary increments (si-processes). Since there is an increasing interest in general Gaussian si-processes as building blocks for models (cf. e.g. [1,15,17,4]), it seems quite relevant to explore methods that allow us to understand the structure of such processes better.

The approach we take was pioneered by M.G. Krein in the 1950s. He investigated problems like interpolation and prediction for stationary processes. The central observation of Krein was that there is a 1–1 relationship between symmetric, Borel measures μ on the line satisfying

$$\int \frac{\mu(\mathrm{d}\lambda)}{1+\lambda^2} < \infty \tag{1.2}$$

and differential operators of the form $f \mapsto \mathrm{d}f'/\mathrm{d}m$ associated with a vibrating string with mass distribution m. Roughly speaking, the measure μ associated with a string describes the kinetic energy of the string as it vibrates at different frequencies. When combined with the theory of reproducing kernel Hilbert spaces (RKHSs) of entire functions of de Branges [5], very precise results are obtained concerning the structure of (subspaces of) $L^2(\mu)$. The general Krein–de Branges theory can be found in Dym and McKean [8] and is recalled for convenience in Section 2 of this paper.

The relevance of this theory for the study of second-order si-processes is easily explained. It is a classical fact that for a continuous, centered, second-order si-process $X = (X_t)_{t \geqslant 0}$ there exists a unique symmetric Borel measure μ on the line satisfying (1.2), such that

$$\mathbb{E}X_sX_t = \int_{\mathbb{R}} \frac{(e^{i\lambda t}-1)(e^{-i\lambda s}-1)}{\lambda^2}\,\mu(d\lambda) \tag{1.3}$$

for $s,t \geqslant 0$ (see e.g. [7, Section XI.11]). Using the Krein–de Branges theory we can associate a unique string with μ and obtain results on the analytic structure of $L^2(\mu)$. Using (1.3) again these translate into probabilistic results for the process X.

To obtain concrete results for a specific process, one has to compute the mass distribution m associated with the given measure μ appearing in (1.3). In general this is a nontrivial task. Krein carried it out for spectra with rational densities and some other examples, see Dym and McKean [8, Chapter 6]. For the fBm, the spectral measure μ in (1.3) is given by

$$\mu(d\lambda) = c_H|\lambda|^{1-2H}\,d\lambda,$$

where $c_H = (\Gamma(1+2H)\sin \pi H)/(2\pi)$ (see for instance [26,29]). The string associated with this measure has, as far we know, never been computed, except for the standard Brownian case $H=\frac{1}{2}$. In Section 4.1 of the present paper the string associated with the fBm is identified. We prove it is the infinitely long string with mass distribution $m(x) = C_H x^{(1-H)/H}$, for some explicitly given constant C_H.

In particular, we will see that the mass distribution arising from the fBm is smooth. For such smooth mass distributions we derive a number of results on the structure of $L^2(\mu)$ from the general Krein–de Branges theory in Section 3. Specialized to the fBm case, we rediscover the results first presented in Dzhaparidze and Van Zanten [11] on the structure of the space $\mathscr{L}_T$, defined as the closure in $L^2(\mu)$ of the span of the collection of functions $\{\lambda \mapsto (\exp(i\lambda t)-1)/i\lambda : t \leqslant T\}$. The results show that $\mathscr{L}_T$ is a RKHS and we obtain an explicit expression for the reproducing kernel. Moreover, we construct a Fourier-type transform on $\mathscr{L}_T$ and exhibit an explicit orthogonal basis. Using the isometry (1.3) these results on the structure of $\mathscr{L}_T$ translate into results on the structure of the process X.

The essential difference with Dzhaparidze and Van Zanten [11] is that in the latter paper we took the moving average representation as given, and used it to derive the results for $L^2(\mu)$. In the present paper the order is reversed. In Section 4 we first derive the results on the structure of $L^2(\mu)$ from the general theory of vibrating strings, without assuming the moving average representation. Then in Section 5 we obtain the moving average and series representations of the fBm as easy consequences.

The identification of the string associated with the fBm also sheds some light on the form of frequency domain results for the fBm obtained in previous papers. In Dzhaparidze and Ferreira [9] and Dzhaparidze and Van Zanten [10,11] we observed that Bessel functions play an important role in the frequency domain, but the deeper reason for their appearance remained unclear. The results in Section 4 of the present paper explain what is going on. It turns out that the spectral measure of the fBm is the principle spectral measure of a differential equation whose eigenfunctions are expressed in terms of Bessel functions (see Corollary 4.3). Properties of Bessel functions are used extensively in various proofs below. For background information we refer to the classic Watson [28], or for instance Lebedev [16].

2. Elements of the spectral theory of vibrating strings

In this section, we recall the elements of the spectral theory of vibrating strings relevant to the study of stochastic processes with stationary increments. This theory was developed by M.G. Krein in the 1950s (see e.g. the historical remarks in [14]). It becomes particularly useful for our purposes when combined with the theory of RKHSs of entire functions of de Branges [5]. In Sections 2.1–2.8 we essentially follow the presentation of this material given in Dym and McKean [8, Chapters 5 and 6]. Omitted proofs and further details can be found there.

2.1. Mathematical description of a vibrating string

Consider a number $l \leqslant \infty$ and a nonnegative, right-continuous, nondecreasing function m on the interval $[0, l)$. The number l is interpreted as the length of a string in equilibrium, $x \in [0, l]$ is thought of as a location on the string, $x = 0$ corresponding to the left endpoint and $x = l$ to the right endpoint, and $m(x)$ is interpreted as the mass of the piece of string from the left endpoint up to (and including) the point x. The jump of m at the point $x \geqslant 0$ is denoted by $\Delta m(x) = m(x) - m(x-)$, and $m(0-) = 0$.

From classical mechanics we know that the motion of the vibrating string is described by the solutions $u = u(t, x)$ of the wave equation

$$m' u_{tt} = u_{xx},$$

at least when m is a smooth function. The number $1/\sqrt{m'(x)}$ can be interpreted as the local propagation speed of the travelling wave, in the sense that it takes a wave

$$T(x) = \int_0^x \sqrt{m'(y)}\,\mathrm{d}y \tag{2.1}$$

time units to travel from the point 0 to x.

One way of analyzing the vibrating string is to begin by looking at periodic solutions u of the form

$$u(t, x) = A(x, \lambda)\mathrm{e}^{\mathrm{i}\lambda t},$$

where λ is a fixed frequency and A describes the amplitude. For u of this form the wave equation reduces to the ordinary differential equation

$$A''(x, \lambda) = -\lambda^2 m'(x) A(x, \lambda) \tag{2.2}$$

for the function A. Note that the kinetic energy of the string which vibrates at the frequency λ is given by

$$\frac{1}{2}\int |u_t(t, x)|^2\,\mathrm{d}m(x) = \frac{1}{2}\lambda^2 \|A(\cdot, \lambda)\|^2_{L^2(m)}. \tag{2.3}$$

Starting from this observation it is possible to associate with every string a unique symmetric measure μ on $\mathbb{R}$ which has the property that $\int(1+\lambda^2)^{-1}\,\mu(\mathrm{d}\lambda)<\infty$, and such that $\lambda^2/\mu(\mathrm{d}\lambda)$ can be interpreted as the kinetic energy of the string which vibrates at the frequency λ. In the remainder of this section we recall the construction of this so-called *principal spectral measure* of a string and some additional facts that we need below.

2.2. Differential operator associated with a string

For general, not necessarily smooth mass distribution functions m, the eigenvalue problem (2.2) can be written as

$$\frac{\mathrm{d}A^+}{\mathrm{d}m} = -\lambda^2 A.$$

Here f^+ denotes the right-hand side derivative of the function f and, similarly, f^- is the left derivative. If we restrict the operator $A \mapsto \mathrm{d}A^+/\mathrm{d}m$ to an appropriate domain it becomes a self-adjoint, negative definite, densely defined operator on the Hilbert space $L^2(m) = L^2([0,l],m)$. We will denote this operator by $\mathscr{G}$, and its domain by $\mathscr{D}(\mathscr{G})$.

The first requirement on $f \in \mathscr{D}(\mathscr{G})$ is of course that $\mathscr{G}f = \mathrm{d}f^+/\mathrm{d}m$ exists, by which we mean that we can write

$$f(x) = f(0) + f^-(0)x + \int_0^x \left(\int_{[0,y]} \mathscr{G}f(z)\,\mathrm{d}m(z) \right) \mathrm{d}y \tag{2.4}$$

for $x \in [0,l)$. We think of functions in $\mathscr{D}(\mathscr{G})$ as being defined on the entire real line by putting m constant outside $[0,l]$, so that $f(x) = f(0) + xf^-(0)$ for $x \leqslant 0$ and $f(x) = f(l) + (x-l)f^+(l)$ for $x \geqslant l$ if $l<\infty$.

Secondly, the functions in $\mathscr{D}(\mathscr{G})$ have to satisfy the appropriate boundary conditions. To make $\mathscr{G}$ self-adjoint we need to impose the boundary condition $f^-(0)=0$ at the left endpoint of the string. For a *long* string, meaning that $l+m(l-)=\infty$, this is the only boundary condition and the situation can be summarized as follows, see Dym and McKean [8, Sections 5.1 and 5.2].

Theorem 2.1. *Suppose that $l+m(l-)=\infty$. Then there exists a dense subset $\mathscr{D}(\mathscr{G})$ of $L^2(m)$ such that every $f \in \mathscr{D}(\mathscr{G})$ has left and right derivatives, satisfies $f^-(0)=0$, and the operator $\mathscr{G} : \mathscr{D}(\mathscr{G}) \to L^2(m)$ given by $\mathscr{G}f = \mathrm{d}f^+/\mathrm{d}m$ is well defined, self-adjoint, and negative definite.*

If the string is *short*, i.e. $l+m(l-)<\infty$, we also have to prescribe how the string is tied down at the right end point. On this side of the string there is a continuum of possible boundary conditions, each leading to a self-adjoint, negative definite operator. The condition can be described by introducing an additional *tying constant* $k \in [0,\infty]$ and prescribing that $f(l+k)=0$ for $f \in \mathscr{D}(\mathscr{G})$. Since $f(l+k) = f(l) + kf^+(l)$, this means that $f(l)+kf^+(l)=0$. For $k=\infty$ this should be interpreted as $f^+(l)=0$. The following theorem summarizes the situation for short strings, cf. of Dym and McKean [8, Sections 5.1 and 5.2].

Theorem 2.2. *Suppose that* $l+m(l-)<\infty$ *and* $k\in[0,\infty]$. *Then there exists a dense subset* $\mathscr{D}(\mathscr{G})$ *of* $L^2(m)$ *such that every* $f\in\mathscr{D}(\mathscr{G})$ *has left and right derivatives, satisfies* $f^-(0)=0$ *and* $f(l)+kf^+(l)=0$, *and the operator* $\mathscr{G}:\mathscr{D}(\mathscr{G})\to L^2(m)$ *given by* $\mathscr{G}f=\mathrm{d}f^+/\mathrm{d}m$ *is well defined, self-adjoint, and negative definite.*

Eq. (2.4) shows that for $f\in\mathscr{D}(\mathscr{G})$, it holds that

$$f^+(x)-f^-(x)=\mathscr{G}f(x)\Delta m(x).$$

So, if the mass function m is continuous, every function in $\mathscr{D}(\mathscr{G})$ is differentiable. Moreover, it holds that if m is absolutely continuous, with derivative m', then $\mathscr{G}f=f''/m'$ for all $f\in\mathscr{D}(\mathscr{G})$.

2.3. Solutions of the eigenvalue problem

Since the operator $\mathscr{G}:\mathscr{D}(\mathscr{G})\to L^2(m)$ is self-adjoint and negative definite, its spectrum $\sigma(\mathscr{G})$ is contained in $(-\infty,0]$. Hence, the second-order ordinary differential equation $\mathscr{G}A=-\lambda^2A$ cannot have a solution in $\mathscr{D}(\mathscr{G})$ if λ^2 is not a real, nonnegative number. However, the equation does have solutions for every $\lambda^2\in\mathbb{C}$. Throughout the paper we denote by $A=A(\cdot,\lambda)$ the solution of

$$\mathscr{G}A(\cdot,\lambda)=-\lambda^2A(\cdot,\lambda),\quad A(0,\lambda)=1,\quad A^-(0,\lambda)=0.$$

For λ^2 outside $[0,\infty)$, a complementary solution $D=D(\cdot,\lambda)$ of the equation can be constructed by setting

$$D(x,\lambda)=A(x,\lambda)\int_x^{l+k}\frac{1}{A^2(y,\lambda)}\,\mathrm{d}y.$$

If the string is short the function $1/A^2$ is integrable, so D is well defined. In the case of a long string the integral is to be taken from x to ∞, and it exists as an improper integral. The function $D=D(\cdot,\lambda)$ satisfies

$$\mathscr{G}D(\cdot,\lambda)=-\lambda^2D(\cdot,\lambda),\quad D^-(0,\lambda)=-1.$$

It is constructed in such a way that for the Wronskian of A and D, we have $A^+D-AD^+=A^-D-AD^-=1$.

A third function that will play an important role below is defined by

$$B(x,\lambda)=-\frac{1}{\lambda}A^+(x,\lambda).$$

It satisfies $\mathrm{d}B=\lambda A\,\mathrm{d}m$ and $B(0,\lambda)=\lambda\Delta m(0)$.

2.4. Resolvents of the string operator

For λ^2 outside $[0,\infty)$ the operator $-\lambda^2I-\mathscr{G}$ is invertible, so that the resolvent

$$R_\lambda=(-\lambda^2I-\mathscr{G})^{-1}$$

is well defined. For all λ^2 outside $[0,\infty)$ we have that $R_\lambda:L^2(m)\to L^2(m)$ is a bounded, self-adjoint operator. Its image, which is $\mathscr{D}(\mathscr{G})$, is dense in $L^2(m)$.

The resolvent R_λ is an integral operator with a kernel that can be expressed in terms of the "eigenfunctions" A and D of the operator $\mathscr{G}$ we introduced in the previous subsection. We define

$$r_\lambda(x,y) = \begin{cases} A(x,\lambda)D(y,\lambda) & \text{if } x \leqslant y, \\ A(y,\lambda)D(x,\lambda) & \text{if } x \geqslant y. \end{cases} \tag{2.5}$$

The following result can be found in Dym and McKean [8, Section 5.4].

Theorem 2.3. *For* λ^2 *outside* $[0,\infty)$, *it holds that*

$$R_\lambda f(x) = \int_{[0,l]} r_\lambda(x,y) f(y)\,\mathrm{d}m(y).$$

2.5. *Spectral measure of a string*

If the string is short, so $l + m(l-) < \infty$, the spectrum of the operator $\mathscr{G}$ is $\{-\lambda_n^2 : n = 1,2,\ldots\}$, where $\lambda_1, \lambda_2, \ldots$ are the nonnegative roots of the equation

$$kA^+(l,\lambda) + A(l,\lambda) = 0 \tag{2.6}$$

(as in Section 2.2, this should be read for $k = \infty$ as $A^+(l,\lambda) = 0$, or, equivalently, $B(l,\lambda) = 0$). The corresponding eigenfunctions are the functions $A(\cdot,\lambda_n)$. Now we construct a symmetric measure μ on the real line by putting mass

$$\frac{\pi}{2\|A(\cdot,\lambda_n)\|^2_{L^2(m)}}$$

at the points $\pm\lambda_n$. We remark that μ has a clear physical interpretation. Up to a constant, the mass $\mu(\{\lambda_n\})$ is equal to $\lambda_n^2/K(\lambda_n)$, where $K(\lambda)$ is the kinetic energy of the string which vibrates at the frequency λ (cf. (2.3)). The measure μ is called the *principal spectral measure* of the string. For λ^2 outside $[0,\infty)$, the resolvent kernel r_λ can be expressed in terms of the measure μ and the eigenfunctions of the string:

$$r_\lambda(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{A(x,\omega)A(y,\omega)}{\omega^2 - \lambda^2}\,\mu(\mathrm{d}\omega). \tag{2.7}$$

Note that this implies in particular that we have the integrability property

$$\int_{\mathbb{R}} \frac{\mu(\mathrm{d}\omega)}{1+\omega^2} = \pi r_i(0,0) < \infty.$$

For long strings the spectral measure can be constructed by first cutting the string to make it short, and then letting the cutting point tend to infinity. The resulting measure is then no longer discrete, in general, but the spectral representation (2.7) of the resolvent kernel still holds. Moreover, there is only one measure with this property.

This uniqueness is in fact not hard to see. By definition (2.5) of the resolvent kernel and (2.7) we have

$$D(0, \mathrm{i}b) = r_{\mathrm{i}b}(0,0) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mu(\mathrm{d}\lambda)}{b^2+\lambda^2}$$

for real-valued b. The symmetric measure μ is completely determined by the values of the integrals on the right-hand side for $b \in \mathbb{R}$ (see e.g. [8, p. 16]). Hence, μ is completely determined by the values $D(0, \mathrm{i}b)$ for $b \in \mathbb{R}$.

The complete theorem reads as follows, cf. Dym and McKean [8, Section 5.5].

Theorem 2.4. *For every given string there exists a unique symmetric measure μ on $\mathbb{R}$ such that* (2.7) *holds. Conversely, given a symmetric measure μ on $\mathbb{R}$ such that $\int(1+\lambda^2)^{-1}\,\mu(\mathrm{d}\lambda)<\infty$ there exists a unique string for which* (2.7) *holds true.*

In view of this theorem, the following definition makes sense.

Definition 2.5. The *principal spectral measure* of the string is the unique measure μ which satisfies (2.7).

2.6. Odd and even transforms

Now let μ be the principal spectral measure of the string and let $L^2_{\mathrm{even}}(\mu)$, resp. $L^2_{\mathrm{odd}}(\mu)$, be the space of even, resp. odd, functions in $L^2(\mu)$. The functions A and B defined in Section 2.3 give rise to integral transformations onto these function spaces. The two following theorems summarize their properties, see Dym and McKean [8, Section 5.7].

Theorem 2.6. *The map $\mathscr{T}_{\mathrm{even}} : L^2(m) \to L^2_{\mathrm{even}}(\mu)$ defined by*

$$\mathscr{T}_{\mathrm{even}} f(\lambda) = \int_{[0,l]} A(x,\lambda) f(x)\,\mathrm{d}m(x)$$

is one to one and onto. Its inverse is given by

$$\mathscr{T}^{-1}_{\mathrm{even}}\psi(x) = \frac{1}{\pi}\int_{\mathbb{R}} A(x,\lambda)\psi(\lambda)\,\mu(\mathrm{d}\lambda).$$

It holds that $\|\mathscr{T}_{\mathrm{even}} f\|^2_{L^2(\mu)} = \pi \|f\|^2_{L^2(m)}$.

So, up to a factor $\sqrt{\pi}$, the map $\mathscr{T}_{\mathrm{even}}$ is a Hilbert space isometry between $L^2(m)$ and $L^2_{\mathrm{even}}(\mu)$. As usual in this kind of setting, the integrals in the statement of the theorem should be interpreted in the wide sense: for short strings they converge as ordinary Lebesgue integrals, but for long strings they may only converge in an L^2-sense.

The domain of the corresponding map into $L^2_{\mathrm{odd}}(\mu)$ is a subspace $\mathscr{X}$ of $L^2([0, l+k], \mathrm{d}x)$, where k is the tying constant if the string is short, and $k=0$ if it is long. For $k<\infty$, $\mathscr{X}$ is defined as the space of functions in $L^2([0, l+k], \mathrm{d}x)$ which are constant on mass-free intervals. For $k=\infty$ we require in addition that the functions vanish on $[l, \infty]$.

Theorem 2.7. *The map* $\mathscr{T}_{\text{odd}} : \mathscr{X} \to L^2_{\text{odd}}(\mu)$ *defined by*

$$\mathscr{T}_{\text{odd}} f(\lambda) = \int_0^{l+k} B(x,\lambda) f(x)\,\mathrm{d}x$$

is one to one and onto. Its inverse is given by

$$\mathscr{T}^{-1}_{\text{odd}} \psi(x) = \frac{1}{\pi} \int_{\mathbb{R}} B(x,\lambda)\psi(\lambda)\,\mu(\mathrm{d}\lambda).$$

It holds that $\|\mathscr{T}_{\text{odd}} f\|^2_{L^2(\mu)} = \pi \|f\|^2_{L^2(\mathbb{R})}$.

If the spectral measure μ has an atom at ω (and hence also at $-\omega$), Theorem 2.6 states that

$$\mathscr{T}^{-1}_{\text{even}}(1_{\{-\omega\}} + 1_{\{\omega\}}) = \frac{2}{\pi} A(\cdot,\omega)\mu(\{\omega\}).$$

Hence, under the even transform the function $1_{\{-\omega\}} + 1_{\{\omega\}} \in L^2_{\text{even}}(\mu)$ corresponds to a function in $L^2(m)$ describing a vibration at frequency ω, with weight, or power, $2\mu(\{\omega\})/\pi$. In general, the theorem allows us to think of an arbitrary element $\psi \in L^2(\mu)$ as a frequency domain description of a vibration of the string, $2\psi(\lambda)\mu(\mathrm{d}\lambda)/\pi$ being the contribution of frequency λ.

Note that if we apply the Parseval relation $\|\mathscr{T}_{\text{even}} f\|^2_{L^2(\mu)} = \pi \|f\|^2_{L^2(m)}$ for the even transform with $f = 1_{\{x\}}$ we see that for all x,

$$\int_{\mathbb{R}} A^2(x,\lambda)\,\mu(\mathrm{d}\lambda) = \frac{\pi}{\Delta m(x)}.$$

In particular, it holds that the principal spectral measure μ has finite total mass if and only if $\Delta m(0) > 0$.

2.7. RKHS of entire functions associated with a string

We now come to the description of the structure of $L^2(\mu)$. The central result is that for short strings, this space is a RKHS of entire functions of the type studied by de Branges [5]. The reproducing kernel can be expressed in terms of the functions A and B introduced in Section 2.3.

Throughout this section, we consider a string with length l, mass distribution m and tying constant k. The following result is proved in Dym and McKean [8, Section 6.3].

Theorem 2.8. *Suppose that the string is short, so* $l + m(l-) < \infty$. *Then* $L^2(\mu)$ *is a RKHS of entire functions. The reproducing kernel is given by*

$$K(\omega,\lambda) = \frac{A(l+k',\omega)B(l,\lambda) - B(l,\omega)A(l+k',\lambda)}{\pi(\lambda-\omega)}, \tag{2.8}$$

where $k' = k$ *if* $k < \infty$ *and* $k' = 0$ *if* $k = \infty$.

Some remarks are in order regarding this theorem. First of all, the elements of $L^2(\mu)$ are equivalence classes of functions. To say that $L^2(\mu)$ is a space of entire functions means that every element admits an entire version. If we consider such an element $\psi \in L^2(\mu)$ in the remainder of the paper, we will always assume it is the smooth version. Secondly, the evaluation of the function A on the right of l should be interpreted as explained in Section 2.1, i.e.

$$A(l+k',\lambda) = A(l,\lambda) + k'A^+(l,\lambda) = A(l,\lambda) - k'\lambda B(l,\lambda).$$

The fact that $K(\omega,\lambda)$ is the reproducing kernel means that $K(\omega,\cdot) \in L^2(\mu)$ for all $\omega \in \mathbb{R}$ and for $\psi \in L^2(\mu)$,

$$\int \psi(\lambda)\overline{K(\omega,\lambda)}\,\mu(\mathrm{d}\lambda) = \langle \psi, K(\omega,\cdot)\rangle_{L^2(\mu)} = \psi(\omega).$$

Finally, we note that expression (2.8) of course only makes sense for $\omega \neq \lambda$. Since $\lambda \mapsto K(\omega,\lambda)$ is smooth, however, $K(\omega,\omega)$ is given by the limit for $\lambda \to \omega$ of the right-hand side of (2.8). Observe that $K(\omega,\omega) = \langle K(\omega,\cdot), K(\omega,\cdot)\rangle_{L^2(\mu)} \geqslant 0$.

For long strings, the situation is more complicated and we can only describe the analytic structure of certain subspaces of $L^2(\mu)$. For $T>0$ we define $L^2_T(\mu)$ as the closure in $L^2(\mu)$ of the linear span of the collection of functions

$$\left\{\lambda \mapsto \frac{\mathrm{e}^{\mathrm{i}\lambda t}-1}{\mathrm{i}\lambda} : |t| \leqslant T\right\}.$$

At this point it is useful to introduce some additional notation. Note that $(\mathrm{e}^{\mathrm{i}\lambda t}-1)/\mathrm{i}\lambda = \hat{1}_{(0,t]}(\lambda)$, where 1_A is the indicator function of the set A and $\hat{f}$ denotes the Fourier transform of the function f, defined by

$$\hat{f}(\lambda) = \int \mathrm{e}^{\mathrm{i}u\lambda} f(u)\,\mathrm{d}u.$$

If V is a linear space and W is a subset of V, we denote the span of W by $\mathrm{sp}\,W$. The closure of this span in V is denoted by $\overline{\mathrm{sp}}\,W$. Using these notations, we have $L^2_T(\mu) = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : |t| \leqslant T\}$.

As the next theorem explains, the functions in $L^2_T(\mu)$ correspond to motions of a finite piece of the string. For $T>0$, we define $x(T)$ as the smallest root $x \geqslant 0$ of the equation

$$T = \int_0^x \sqrt{m'(y)}\,\mathrm{d}y, \tag{2.9}$$

where m' is the derivative of the absolute continuous part of m. Note that the function $x(T)$ is the inverse of the function $T(x)$ defined by (2.1). Hence, a wave starting at location $x=0$ at time zero arrives at location $x(T)$ at time T.

Theorem 2.9. *Suppose that* $0 < T < \int_0^l \sqrt{m'(y)}\,\mathrm{d}y$. *Then* $L^2_T(\mu) = L^2(\mu_T)$, *where* μ_T *is the principal spectral measure of the string with length* $x(T)$, *the same mass distribution* m, *and tying constant* $k=\infty$. *In particular,* $L^2_T(\mu)$ *is a RKHS of entire functions and its reproducing kernel is given by*

$$K_T(\omega,\lambda) = \frac{A(x(T),\omega)B(x(T),\lambda) - B(x(T),\omega)A(x(T),\lambda)}{\pi(\lambda-\omega)}. \tag{2.10}$$

Proof. The theorem follows from a combination of results of Dym and McKean [8]. Firstly, it is stated on p. 241 that $L_T(\mu)$ (Z^T in the notation of the book) equals the span of the class of functions in $L^2(\mu)$ of exponential type strictly smaller than T. Under the condition of the theorem, the latter space is identified on p. 244 as the so-called Krein space $\mathbb{K}^-$ attached to $x(T)$. By the amplification on p. 236 this space is the de Brange space associated with the function $E(\lambda) = A(x(T), \lambda) - \mathrm{i}B(x(T), \lambda)$. This is a RKHS of entire functions which has (2.10) as reproducing kernel. $\square$

2.8. Orthogonal basis of the RKHS

By combining a number of the result recalled thus far, we can obtain an explicit orthonormal basis of the space $L^2_T(\mu) = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : |t| \leqslant T\}$. This idea is explained very briefly in Dym and McKean [8, Section 6.11]. For completeness we include a proof of the following result.

Theorem 2.10. *Suppose that* $0 < T < \int_0^l \sqrt{m'(y)}\,\mathrm{d}y$. *Let* $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ *be the real-valued zeros of the function* $\omega \mapsto B(x(T), \omega)$. *The collection* $\{K_T(\omega_n, \cdot) : n \in \mathbb{Z}\}$ *is an orthogonal basis of* $L^2_T(\mu)$. *Each function* $\psi \in L^2_T(\mu)$ *can be expanded as*

$$\psi(\lambda) = \sum_{n \in \mathbb{Z}} \psi(\omega_n) \frac{K_T(\omega_n, \lambda)}{K_T(\omega_n, \omega_n)},$$

the convergence taking place in $L^2(\mu)$.

Proof. For $n, m \in \mathbb{Z}$ we have, by the reproducing property of K_T,

$$\langle K_T(\omega_n, \cdot), K_T(\omega_m, \cdot)\rangle_\mu = K_T(\omega_n, \omega_m).$$

Hence $\|K_T(\omega_n, \cdot)\|^2_\mu = K_T(\omega_n, \omega_n)$, and expression (2.10) implies that the functions are orthogonal if $n \neq m$.

We have that $L^2_T(\mu) = L^2(\mu_T)$, where μ_T is the spectral measure of the string with mass m, length $x(T)$ and tying constant $k = \infty$. As explained in Section 2.5, this means that μ_T is a measure which only has isolated masses, located at the zeros ω_n of $B(x(T), \cdot)$. Hence, a function $\psi \in L^2_T(\mu) = L^2(\mu_T)$ vanishes if it vanishes at every zero ω_n. Now suppose that ψ is orthogonal to every $K_T(\omega_n, \cdot)$. Then by the reproducing property, $\psi(\omega_n) = 0$ for every $n \in \mathbb{Z}$, hence ψ vanishes in $L^2_T(\mu)$. This shows that the functions $K_T(\omega_n, \cdot)$ form a complete system.

Since the $K_T(\omega_n, \cdot)$ form a complete orthogonal system, we have

$$\psi(\lambda) = \sum_n \frac{\langle \psi, K_T(\omega_n, \cdot)\rangle_\mu}{\|K_T(\omega_n, \cdot)\|^2_\mu} K_T(\omega_n, \lambda)$$

for $\psi \in L^2_T(\mu)$. By the reproducing property the inner product appearing in the sum equals $\psi(\omega_n)$. The squared norm was just shown to be $K_T(\omega_n, \omega_n)$. $\square$

Let us remark here that it is also possible to produce different orthogonal bases of $L^2_T(\mu)$, using the zeros of equation (2.6) for different $k \in [0, \infty)$. See Section 6.11 of Dym and McKean [8]. Theorem 2.10 deals with the case $k = \infty$.

3. RKHS associated with a smooth string

Given a spectral measure μ, Theorem 2.9 describes the RKHS structure of the space $L^2_T(\mu) = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : t \in [-T, T]\} \subseteq L^2(\mu)$. In connection with our study of stochastic processes, however, we are more interested in the space

$$\mathscr{L}_T = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : t \in [0, T]\} \subseteq L^2(\mu).$$

From the simple observation that if $\psi \in \mathscr{L}_{2T}$, then $\lambda \mapsto \exp(-\mathrm{i}\lambda T)\psi(\lambda)$ belongs to $L^2_T(\mu)$, it easily follows that the reproducing kernel on $\mathscr{L}_{2T}$ is given by

$$S_{2T}(\omega, \lambda) = \mathrm{e}^{\mathrm{i}(\lambda-\omega)T} K_T(\omega, \lambda). \tag{3.1}$$

In this section, we study the structure of the RKHS $\mathscr{L}_T$ in the case that the mass distribution function m is smooth. We first derive an integral representation of the reproducing kernel and use it to define a Fourier-type transform on $\mathscr{L}_T$. In addition we obtain an orthogonal basis of the space $\mathscr{L}_T$.

So far, in the presentation of the classical results in Section 2, we either considered functions of the variable λ (or ω), which denotes a frequency, or functions of the variable x (or y), which denotes a location on the string. For our present purposes it is necessary to replace the location variable x by the time variable t. This change of variables appeared already implicitly in the preceding section, where we introduced the inverse $x(T)$ of the function $T(x)$ defined by (2.1).

In this section, we suppose that the mass distribution function m is continuously differentiable, its derivative m' is strictly positive and we have the finite propagation speed property

$$\int_0^l \sqrt{m'(y)}\, \mathrm{d}y = \infty.$$

Observe that the map $t \mapsto x(t)$ is then differentiable, whence we can define the functions a and b by

$$a(t, \lambda) = A(x(t), \lambda), \quad b(t, \lambda) = B(x(t), \lambda)x'(t). \tag{3.2}$$

Next, the functions φ and V are defined by

$$\varphi(2t, \lambda) = \mathrm{e}^{\mathrm{i}\lambda t}(a(t, \lambda) + \mathrm{i}b(t, \lambda)) \tag{3.3}$$

and

$$V(2t) = \frac{1}{\pi} m(x(t)). \tag{3.4}$$

Note that the assumptions on m imply that V is smooth, strictly increasing, and $V(0) = 0$.

The following theorem gives an integral representation for the reproducing kernel of $\mathscr{L}_T$.

Theorem 3.1. *The reproducing kernel of $\mathscr{L}_T$ can be written as*

$$S_T(\omega,\lambda)=\int_0^T \varphi(t,\lambda)\overline{\varphi(t,\omega)}\,\mathrm{d}V(t). \tag{3.5}$$

Proof. Note that (3.1) implies that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{2t}(\omega,\lambda)=\mathrm{e}^{\mathrm{i}(\lambda-\omega)t}\left(\frac{\mathrm{d}}{\mathrm{d}t}K_t(\omega,\lambda)+\mathrm{i}(\lambda-\omega)K_t(\omega,\lambda)\right).$$

By Theorem 2.9, it follows from a straightforward calculation that

$$\pi\frac{\mathrm{d}}{\mathrm{d}t}S_{2t}(\omega,\lambda)=\varphi(t,\lambda)\overline{\varphi(t,\omega)}\frac{\mathrm{d}}{\mathrm{d}t}m(x(t)).$$

Integration of this identity completes the proof. $\square$

Observe that since $\varphi(0,\lambda)=1$, it follows from (3.5) that

$$S_t(0,\lambda)=\int_0^t \varphi(u,\lambda)\,\mathrm{d}V(u). \tag{3.6}$$

In particular, we have $S_t(0,0)=V_t$.

Corollary 3.2. *The subspaces* $\mathrm{sp}\{S_T(\omega,\cdot):\omega\in\mathbb{R}\}$ *and* $\mathrm{sp}\{S_t(0,\cdot):t\in[0,T]\}$ *are dense in* $\mathscr{L}_T$.

Proof. Suppose that $\psi\in\mathscr{L}_T$ is orthogonal to all functions $S_T(\omega,\cdot)$. Then for every $\omega\in\mathbb{R}$

$$\psi(\omega)=\int\psi(\lambda)\overline{S_T(\omega,\lambda)}\,\mu(\mathrm{d}\lambda)=0,$$

hence ψ vanishes. This shows that the first subspace is dense. To prove that the second one is dense it now suffices to show that for every $\omega\in\mathbb{R}$, the function $S_T(\omega,\cdot)$ belongs to $\overline{\mathrm{sp}}\{S_t(0,\cdot):t\in[0,T]\}$. It follows from (3.6) that

$$S_T(\omega,\lambda)=\int_0^T \varphi(t,\lambda)\overline{\varphi(t,\omega)}\,\mathrm{d}V(t)=\int_0^T \overline{\varphi(t,\omega)}\,\mathrm{d}S_t(0,\lambda).$$

This completes the proof. $\square$

Using the result of the preceding theorem we can now introduce a Fourier-type transform between the spaces $L^2([0,T],V)$ and $\mathscr{L}_T$, the function $\varphi(t,\lambda)$ acting as the Fourier kernel. By putting $\omega=\lambda$ in (3.5) we see that $\varphi(\cdot,\lambda)\in L^2([0,T],V)$, so for $f\in L^2([0,T],V)$ the function

$$\mathscr{U}f(\lambda)=\int_0^T f(t)\varphi(t,\lambda)\,\mathrm{d}V(t)$$

is well defined, by Cauchy–Schwarz. We have the following result regarding the linear operator $\mathscr{U}$.

Theorem 3.3. *We have that* $\mathscr{U} : L^2([0,T],V) \to \mathscr{L}_T$ *is an isometry and*

$$\mathscr{U}^{-1}\psi(t) = \frac{\mathrm{d}}{\mathrm{d}V(t)} \int \psi(\lambda)\left(\int_0^t \overline{\varphi(u,\lambda)}\,\mathrm{d}V(u)\right)\mu(\mathrm{d}\lambda) \tag{3.7}$$

for $\psi \in \mathscr{L}_T$.

Proof. By (3.5) the operator $\mathscr{U}$ maps the indicator function $1_{(0,t]}$ to $S_t(0,\cdot) \in \mathscr{L}_t \subseteq \mathscr{L}_T$. Hence, by linearity, the simple functions in $L^2([0,T],V)$ are mapped to $\mathrm{sp}\{S_t(0,\cdot) : t \in [0,T]\} \subseteq \mathscr{L}_T$. Moreover, using the preceding theorem it is straightforward to verify that $\mathscr{U}$ is an isometry between these two subspaces. Since the simple functions are dense in $L^2([0,T],V)$ and $\mathrm{sp}\{S_t(0,\cdot) : t \in [0,T]\}$ is dense in $\mathscr{L}_T$, we see that $\mathscr{U}$ is indeed an isometry between $L^2([0,T],V)$ and $\mathscr{L}_T$.

Since $\mathscr{U}$ is an isometry, its inverse $\mathscr{U}^{-1}$ coincides with its adjoint $\mathscr{U}^*$. For the adjoint we have

$$\begin{aligned}\int_0^t \mathscr{U}^*\psi(u)\,\mathrm{d}V(u) &= \langle \mathscr{U}^*\psi, 1_{(0,t]}\rangle_V \\ &= \langle \psi, \mathscr{U}1_{(0,t]}\rangle_\mu \\ &= \int \psi(\lambda)\overline{S_t(0,\lambda)}\,\mu(\mathrm{d}\lambda)\end{aligned}$$

for every $t \geqslant 0$. By differentiating with respect to t and using (3.6) we obtain the formula for $\mathscr{U}^{-1}$. $\square$

Observe that if the function $\psi \in \mathscr{L}_T$ is such that the order of the differentiation and integration operations in (3.7) may be reversed, the inversion formula reduces to

$$\mathscr{U}^{-1}\psi(t) = \int \psi(\lambda)\overline{\varphi(t,\lambda)}\,\mu(\mathrm{d}\lambda). \tag{3.8}$$

In general, however, this integral on the right-hand side does not converge as a Lebesgue integral, but only in an L^2-sense.

We have an explicit expression for the reproducing kernel S_T in the spirit of Theorem 2.9.

Theorem 3.4. *We have*

$$S_T(\omega,\lambda) = 2V'(T)\exp\left(\mathrm{i}\,\frac{(\lambda-\omega)T}{2}\right)\frac{a(\frac{T}{2},\omega)b(\frac{T}{2},\lambda) - b(\frac{T}{2},\omega)a(\frac{T}{2},\lambda)}{\lambda-\omega}.$$

Proof. Observe that by Theorem 2.9,

$$K_t(\omega,\lambda) = \frac{a(t,\omega)b(t,\lambda) - b(t,\omega)a(t,\lambda)}{\pi x'(t)(\lambda-\omega)}.$$

Differentiation of the identity

$$t = \int_0^{x(t)} \sqrt{m'(y)}\,\mathrm{d}y$$

shows that

$$2V'(2t) = \frac{\mathrm{d}}{\mathrm{d}t} V(2t) = \frac{1}{\pi} m'(x(t))x'(t) = \frac{1}{\pi x'(t)}.$$

Combination of these expressions with (3.1) yields the theorem. $\square$

Using the results of Section 2.8, we easily obtain an orthogonal basis of $\mathscr{L}_T$ and the corresponding sampling formula.

Theorem 3.5. *Let $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ be the real-valued zeros of the function $b(T/2, \cdot)$. The collection $\{S_T(\omega_n, \cdot) : n \in \mathbb{Z}\}$ is an orthogonal basis of $\mathscr{L}_T$. Each function $\psi \in \mathscr{L}_T$ can be expanded as*

$$\psi(\lambda) = \sum_{n \in \mathbb{Z}} \psi(\omega_n) \frac{S_T(\omega_n, \lambda)}{S_T(\omega_n, \omega_n)}.$$

Proof. Follows from Theorem 2.10 and (3.1). $\square$

4. String and RKHS associated with the fBm

4.1. The string associated with the fBm

In this section, we identify the string associated with the fBm with Hurst index $H \in (0, 1)$. We prove that the spectral measure μ_H of the fBm is the principal spectral measure of an infinitely long string with mass distribution $C_H x^{(1-H)/H}$, for an explicitly given constant C_H, and we give explicit expressions for the eigenfunctions A, B and D introduced in Section 2.3. We will see in particular that the string associated with ordinary Brownian motion is the infinitely long string with mass distribution $m(x) = 4\pi^2 x$. This is in fact a known result, cf. e.g. Dym and McKean [8, Chapter 5]. The results obtained for $H \neq 1/2$ are new.

We begin with the observation that just from scaling properties, it follows that the principal spectral measure of an infinitely long string with power mass function is an absolutely continuous measure with a power density, and the two powers can be related explicitly. In probabilistic terms: we will see that it follows from the H-self similarity of the fBm that the corresponding mass function is some constant times $x \mapsto x^{(1-H)/H}$.

Theorem 4.1. *The principal spectral measure of an infinitely long string with mass distribution function $m(x) = cx^p$ for $c, p > 0$ is absolutely continuous, and its density is given by $\lambda \mapsto C|\lambda|^{(p-1)/(p+1)}$ for some $C > 0$.*

Proof. The eigenfunction A of the string solves the eigenvalue problem

$$A''(x, \lambda) = -\lambda^2 m'(x) A(x, \lambda)$$

(see Section 2.3). Now fix $a > 0$ and define the function $F(x, \lambda) = A(ax, \lambda)$. Since $m'(ax) = a^{p-1} m'(x)$, this function satisfies

$$F''(x, \lambda) = -\left(\lambda a^{\frac{p+1}{2}}\right)^2 m'(x) F(x, \lambda).$$

We see that F satisfies the same equation as $A(x, a^{(p+1)/2}\lambda)$. Hence, since it also satisfies the same initial conditions $F(0,\lambda) = 1$ and $F'(0,\lambda) = 0$, we have $F(x,\lambda) = A(x, a^{(p+1)/2}\lambda)$. In other words, the eigenfunctions have the property that $A(ax,\lambda) = A(x, a^{(p+1)/2}\lambda)$, or $A(x, a\lambda) = A(a^{2/(p+1)}x, \lambda)$. Differentiation of the latter identity yields the relation $B(x, a\lambda) = a^{(1-p)/(1+p)}B(a^{2/(1+p)}x, \lambda)$ for the function B defined in Section 2.3.

These scaling properties of the functions A and B carry over to the odd and even transforms introduced in Section 2.6. For $f \in L^2(m)$ we have the relation

$$\begin{aligned}
\mathscr{T}_{\text{even}}f(a\lambda) &= \int_0^\infty A(x, a\lambda)f(x)\,\mathrm{d}m(x) \\
&= \int_0^\infty A\left(a^{\frac{2}{1+p}}x, \lambda\right)f(x)\,\mathrm{d}m(x) \\
&= a^{\frac{-2p}{1+p}}\int_0^\infty A(x,\lambda)f\left(a^{-2/(1+p)}x\right)\,\mathrm{d}m(x) \\
&= a^{\frac{-2p}{1+p}}\mathscr{T}_{\text{even}}\tilde{f}(\lambda),
\end{aligned}$$

where $\tilde{f}(x) = f(a^{-2/(1+p)}x)$. Since $\|\tilde{f}\|^2_{L^2(m)} = a^{2p/(1+p)}\|f\|^2_{L^2(m)}$, it follows that

$$\begin{aligned}
\|\mathscr{T}_{\text{even}}f(a\cdot)\|^2_{L^2(\mu)} &= a^{\frac{-4p}{1+p}}\|\mathscr{T}_{\text{even}}\tilde{f}\|^2_{L^2(\mu)} \\
&= \pi a^{\frac{-4p}{1+p}}\|\tilde{f}\|^2_{L^2(m)} \\
&= \pi a^{\frac{-2p}{1+p}}\|f\|^2_{L^2(m)} \\
&= a^{\frac{-2p}{1+p}}\|\mathscr{T}_{\text{even}}f\|^2_{L^2(\mu)}.
\end{aligned}$$

Similar calculations for the odd transform imply that for all $f \in L^2[0,\infty)$,

$$\|\mathscr{T}_{\text{odd}}f(a\cdot)\|^2_{L^2(\mu)} = a^{-2p/(1+p)}\|\mathscr{T}_{\text{odd}}f\|^2_{L^2(\mu)}.$$

Consequently, we have the scaling property

$$\|\psi(a\cdot)\|_{L^2(\mu)} = a^{-p/(1+p)}\|\psi\|_{L^2(\mu)}$$

for every $\psi \in L^2(\mu)$.

If we take $\lambda \in \mathbb{R}$ and $\psi = 1_{\{\lambda\}}$ we find that for $a > 0$,

$$\mu(\{\lambda/a\}) = \|\psi(a\cdot)\|^2_{L^2(\mu)} = a^{-2p/(1+p)}\|\psi\|^2_{L^2(\mu)} = a^{-2p/(1+p)}\mu(\{\lambda\}),$$

which implies that μ does not have atoms. Similarly, taking $\psi = 1_{(0,1]}$ implies that

$$\mu(0,\lambda] = \lambda^{\frac{2p}{1+p}}\mu(0,1]$$

for all $\lambda > 0$. This completes the proof. $\square$

The preceding theorem shows that for the mass distribution $m(x) = x^{(1-H)/H}$ for $H \in (0,1)$, the principle spectral measure is given by $\mu(\mathrm{d}\lambda) = C|\lambda|^{1-2H}\,\mathrm{d}\lambda$ for some constant $C > 0$. The following theorem provides the value of the constant and gives explicit expressions for the functions A, B and D introduced in Section 2.3. The proof of the theorem relies heavily on properties of Bessel functions. We refer to Watson [28] or Lebedev [16] for background.

Theorem 4.2. *Let $H \in (0,1)$ be given. The measure*

$$\mu(\mathrm{d}\lambda) = \frac{\pi H 4^H}{\Gamma^2(1-H)}\,|\lambda|^{1-2H}\,\mathrm{d}\lambda$$

is the principal spectral measure of the infinitely long string with mass distribution

$$m(x) = \frac{x^{\frac{1-H}{H}}}{4H(1-H)}.$$

The corresponding eigenfunctions are given by

$$A(x,\lambda) = \Gamma(1-H)\left(\frac{\lambda}{2}\right)^H \sqrt{x}J_{-H}(\lambda x^{\frac{1}{2H}}),$$

and

$$B(x,\lambda) = -\frac{1}{\lambda}\,A'(x,\lambda) = \frac{\Gamma(1-H)}{2H}\left(\frac{\lambda}{2}\right)^H x^{\frac{1-H}{2H}} J_{1-H}(\lambda x^{\frac{1}{2H}}).$$

For $b \in \mathbb{R}$ we have

$$D(x,\mathrm{i}b) = \frac{2H}{\Gamma(1-H)}\left(\frac{2}{b}\right)^H \sqrt{x}K_{-H}(bx^{\frac{1}{2H}}).$$

Here J_ν is the Bessel function of the first of order ν and K_ν is the modified Bessel function of the second kind of order ν.

Proof. If $m(x) = x^{\frac{1-H}{H}}/(4H(1-H))$, then

$$m'(x) = \frac{1}{4H^2}\,x^{\frac{1-2H}{H}}$$

and hence the equations for the function $A = A(\cdot,\lambda)$ are given by

$$A'' + \frac{\lambda^2}{4H^2}\,x^{\frac{1-2H}{H}}A = 0, \quad A(0) = 1, \quad A'(0) = 0. \tag{4.1}$$

According to Gradshteyn and Ryzhik [12, Formula 4, Table 8.491, p. 971] (applied with $\beta = \lambda, \gamma = 1/(2H)$ and $\nu = -H$) the differential equation is solved by the function

$$A(x) = c\sqrt{x}J_{-H}(\lambda x^{\frac{1}{2H}}),$$

where c is an arbitrary constant. Using the power series expansion of the Bessel function it is easily verified that this function also satisfies $A'(0) = 0$, as required.

Moreover, from the power series we see that

$$A(0) = c\,\frac{2^H}{\lambda^H \Gamma(1-H)}.$$

Hence, the requirement $A(0) = 1$ leads to the choice $c = \lambda^H \Gamma(1-H)/2^H$ and we find that for this string, the eigenfunctions are given by

$$A(x,\lambda) = \Gamma(1-H)\left(\frac{\lambda}{2}\right)^H \sqrt{x} J_{-H}(\lambda x^{\frac{1}{2H}}).$$

The expression for B is obtained by using the formula

$$\frac{\mathrm{d}}{\mathrm{d}z} J_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z)$$

(cf. Watson [28, p. 45]) for the Bessel function of the first kind.

To derive the expression for D, observe first that for $\lambda = \mathrm{i}b$, $b \in \mathbb{R}$, the solution $A(\cdot, \mathrm{i}b)$ of (4.1) is given by

$$\begin{aligned} A(x,\mathrm{i}b) &= \Gamma(1-H)\left(\frac{b}{2}\right)^H \sqrt{x} \mathrm{e}^{\mathrm{i}\frac{\pi H}{2})} J_{-H}(\mathrm{i}b x^{\frac{1}{2H}}) \\ &= \Gamma(1-H)\left(\frac{b}{2}\right)^H \sqrt{x} I_{-H}(b x^{\frac{1}{2H}}), \end{aligned}$$

where I_ν is the modified Bessel function of the first kind of order ν. Denoting the modified Bessel function of the second kind of order ν by K_ν, we have the Wronskian relation $I_\nu(z)K_\nu'(z) - K_\nu(z)I_\nu'(z) = -1/z$ (see [16, Formula (5.9.5)]). It follows that for $x \leqslant y$

$$\int_x^y \frac{\mathrm{d}z}{z^2 I_\nu^2(z)} = \frac{K_\nu(x)}{I_\nu(x)} - \frac{K_\nu(y)}{I_\nu(y)}$$

and hence, by the asymptotic properties of K_ν and I_ν (see [16, Formula (5.16.5)]),

$$\int_x^\infty \frac{\mathrm{d}z}{z^2 I_\nu^2(z)} = \frac{K_\nu(x)}{I_\nu(x)}.$$

A straightforward computation now shows that

$$D(x,\mathrm{i}b) = A(x,\mathrm{i}b)\int_x^\infty \frac{\mathrm{d}y}{A^2(y,\mathrm{i}b)} = \frac{2H}{\Gamma(1-H)}\left(\frac{2}{b}\right)^H \sqrt{x} K_{-H}(b x^{\frac{1}{2H}}).$$

It is argued in Section 2.5 that for the proof of the formula for μ, it suffices to show that

$$D(0,\mathrm{i}b) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mu(\mathrm{d}\lambda)}{b^2 + \lambda^2}$$

for every $b \in \mathbb{R}$. By Gradshteyn and Ryzhik [12, Formula 3.251.2, p. 295] (applied with $v = 0$ and $\mu = 2 - 2H$) the right-hand side equals

$$\frac{H4^H \Gamma(H)}{b^{2H}\Gamma(1-H)}.$$

Using Lebedev [16, Formulas (5.16.4) and (5.7.10)], we see that this indeed coincides with $D(0, \mathrm{i}b)$. $\square$

Recall that the covariance function of an fBm X with Hurst index $H \in (0,1)$ is given by

$$\begin{aligned}\mathbb{E}X_sX_t &= \tfrac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}) \\ &= \int_{\mathbb{R}} \hat{1}_{(0,t]}(\lambda)\overline{\hat{1}_{(0,s]}(\lambda)}\, \mu_H(\mathrm{d}\lambda),\end{aligned}$$

where μ_H is the spectral measure of the fBm, given by $\mu_H(\mathrm{d}\lambda) = c_H|\lambda|^{1-2H}\,\mathrm{d}\lambda$, with

$$c_H = \frac{\Gamma(1+2H)\sin \pi H}{2\pi}$$

(cf. e.g. [29,26]). The preceding theorem yields the following result for μ_H.

Corollary 4.3. *The measure μ_H is the principle spectral measure of the infinitely long string with mass distribution*

$$m(x) = \frac{\kappa_H^{1/H}}{4H(1-H)}\, x^{\frac{1-H}{H}},$$

where

$$\kappa_H = \frac{2\pi^{3/2}}{\Gamma(1-H)\Gamma(1/2+H)} = \frac{2\sqrt{\pi}\Gamma(H)\sin \pi H}{\Gamma(1/2+H)}.$$

The associated eigenfunctions are given by

$$A(x,\lambda) = \Gamma(1-H)\left(\frac{\lambda}{2}\right)^H \sqrt{\kappa_H x}J_{-H}(\lambda(\kappa_H x)^{\frac{1}{2H}})$$

and

$$B(x,\lambda) = \frac{\kappa_H\Gamma(1-H)}{2H}\left(\frac{\lambda}{2}\right)^H (\kappa_H x)^{\frac{1-H}{2H}} J_{1-H}(\lambda(\kappa_H x)^{\frac{1}{2H}}).$$

For $b \in \mathbb{R}$ it holds that

$$D(x, \mathrm{i}b) = \frac{2H}{\Gamma(1-H)\kappa_H}\left(\frac{2}{b}\right)^H \sqrt{\kappa_H x}K_{-H}(b(\kappa_H x)^{\frac{1}{2H}}).$$

Proof. It is easy to see that if the measure μ is the principal spectral measure of the infinitely long string with mass distribution m, then for $c>0$, the measure $c\mu$ corresponds to the infinitely long string with mass distribution $c^{-1}m(c^{-1}x)$, and the functions $A(x,\lambda)$, $B(x,\lambda)$ and $D(x,\lambda)$ have to be replaced by $A(c^{-1}x,\lambda)$, $c^{-1}B(c^{-1}x,\lambda)$ and $cD(c^{-1}x,\lambda)$, respectively (this is "Rule 1" on [8, p. 265]). □

Observe that in the standard Brownian case $H=\frac{1}{2}$ we have $\kappa_{1/2}=2\pi$. So indeed, the string associated with ordinary Brownian motion is the infinitely long string with mass distribution $m(x)=4\pi^2 x$.

4.2. Frequency domain RKHS associated with the fBm

The results of the preceding subsection show that the mass function m of the string associated with the fBm is smooth, and its density is given by

$$m'(x)=\frac{\kappa_H^{1/H}}{4H^2}\,x^{\frac{1-2H}{H}}.$$

This implies in particular that we have the finite propagation speed property

$$\int_0^l \sqrt{m'(y)}\,\mathrm{d}y=\infty,$$

so all conditions posed in Section 3 are satisfied.

To compute the functions a, b, φ and V defined by (3.2), (3.3) and (3.4) in the present case, observe first that the function T defined by (2.1) is now given by

$$T(x)=(\kappa_H x)^{\frac{1}{2H}},$$

so that $\kappa_H x(T)=T^{2H}$. Hence, we obtain

$$a(t,\lambda)=\Gamma(1-H)\left(\frac{\lambda t}{2}\right)^H J_{-H}(\lambda t),$$

$$b(t,\lambda)=\Gamma(1-H)\left(\frac{\lambda t}{2}\right)^H J_{1-H}(\lambda t),$$

$$\varphi(t,\lambda)=\Gamma(1-H)\left(\frac{\lambda t}{4}\right)^H \mathrm{e}^{\mathrm{i}\frac{\lambda t}{2}}\left(J_{-H}\left(\frac{\lambda t}{2}\right)+\mathrm{i}J_{1-H}\left(\frac{\lambda t}{2}\right)\right),$$

$$V(t)=\frac{\sqrt{\pi}}{2^{3-2H}H\Gamma(2-H)\Gamma(1/2+H)}\,t^{2-2H}.$$

The three theorems below are now immediate consequences of the results for general smooth strings obtained in Section 3. Observe that the statements are precisely those of, respectively, Dzhaparidze and Van Zanten [11, Corollary 6.2, Theorems 3.2 and 7.2]. In the latter paper, however, we derived the results by taking the whitening formula and moving average representation of the fBm as the starting point. In the present paper, we argue in the opposite direction. The results for $\mathscr{L}_T$ are derived from general results for smooth strings, and the whitening formula and moving average representation are obtained as corollaries in the next section.

Theorem 4.4. *The space $\mathscr{L}_T$ is a RKHS of entire functions, with reproducing kernel given by*

$$\begin{aligned}&\frac{S_T(\omega,\lambda)}{(4-4H)\Gamma^2(1-H)V_T}\\&\quad=\mathrm{e}^{\mathrm{i}\frac{(\lambda-\omega)T}{2}}\left(\frac{\omega\lambda T^2}{16}\right)^H\frac{J_{-H}(\frac{\omega T}{2})J_{1-H}(\frac{\lambda T}{2})-J_{1-H}(\frac{\omega T}{2})J_{-H}(\frac{\lambda T}{2})}{(\lambda-\omega)T}.\end{aligned}$$

Proof. Follows from Theorem 3.4. □

If we let $\lambda\to\omega$ in the preceding expression for the reproducing kernel S_T and use the recurrence formulae for the Bessel function J_ν (cf. [28, p. 45]) we see that we have

$$\begin{aligned}&\frac{S_T(\omega,\omega)}{(2-2H)\Gamma^2(1-H)V_T}\\&\quad=\left(\frac{\omega T}{4}\right)^{2H}\frac{2}{T}\left(J_{-H}\left(\frac{\omega T}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\omega}J_{1-H}\left(\frac{\omega T}{2}\right)-J_{1-H}\left(\frac{\omega T}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\omega}J_{-H}\left(\frac{\omega T}{2}\right)\right)\\&\quad=\left(\frac{\omega T}{4}\right)^{2H}\left(J^2_{1-H}\left(\frac{\omega T}{2}\right)+\frac{4H-2}{\omega T}J_{-H}\left(\frac{\omega T}{2}\right)J_{1-H}\left(\frac{\omega T}{2}\right)+J^2_{-H}\left(\frac{\omega T}{2}\right)\right)\end{aligned}$$

on the diagonal.

The isometry $\mathscr{U}$ takes the following form in the fBm case.

Theorem 4.5. *We have an isometry $\mathscr{U}:L^2([0,T],V)\to\mathscr{L}_T$, given by*

$$\mathscr{U}f(\lambda)=\Gamma(1-H)\int_0^T f(t)\left(\frac{\lambda t}{4}\right)^H\mathrm{e}^{\mathrm{i}\frac{\lambda t}{2}}\left(J_{-H}\left(\frac{\lambda t}{2}\right)+iJ_{1-H}\left(\frac{\lambda t}{2}\right)\right)\mathrm{d}V(t).$$

Proof. Follows from Theorem 3.3. □

Observe that for $H=\frac{1}{2}$ we have $\varphi(t,\lambda)=\exp(\mathrm{i}\lambda t)$, $V(t)=t$ and $\Gamma(\frac{1}{2})=\sqrt{\pi}$, so $\mathscr{U}$ is simply the Fourier transform in this case.

Theorem 4.6. *Let $\cdots<\omega_{-1}<\omega_0=0<\omega_1<\cdots$ be the real-valued zeros of the Bessel function J_{1-H}. The collection $\{S_T(2\omega_n/T,\cdot):n\in\mathbb{Z}\}$ is an orthogonal basis of $\mathscr{L}_T$. Each function $\psi\in\mathscr{L}_T$ can be expanded as*

$$\psi(\lambda)=\sum_{n\in\mathbb{Z}}\psi(2\omega_n/T)\frac{S_T(2\omega_n/T,\lambda)}{S_T(2\omega_n/T,2\omega_n/T)}.$$

Proof. Follows from Theorem 3.5. □

5. Representations of fractional Brownian motion

Let $X = (X_t)_{t \geqslant 0}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. Say the process is defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and for some fixed time horizon $T \geqslant 0$, define the linear space $\mathscr{H}_T = \overline{\mathrm{sp}}\{X_t : t \in [0, T]\} \subseteq L^2(\mathbb{P})$. The spectral representation

$$\mathbb{E}X_s X_t = \langle \hat{1}_{(0,s]}, \hat{1}_{(0,t]} \rangle_{\mu_H}$$

gives rise to an isometry between $\mathscr{H}_T$ and $\mathscr{L}_T = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : t \in [0, T]\} \subseteq L^2(\mu_H)$, determined by the relation $X_t \longleftrightarrow \hat{1}_{(0,t]}$. We call this map the spectral isometry.

The spectral representation can be used to define a stochastic integral with respect to X of a large class of deterministic integrands. Consider the class of functions $\mathscr{I}_T = \{f \in L^2[0, T] : \hat{f} \in L^2(\mu_H)\}$ and endow it with the inner product $\langle f, g \rangle_{\mathscr{I}_T} = \langle \hat{f}, \hat{g} \rangle_{\mu_H}$. Then the spectral representation can be written as $\mathbb{E}X_s X_t = \langle 1_{(0,s]}, 1_{(0,t]} \rangle_{\mathscr{I}_T}$. In particular, the mapping $1_{(0,t]} \to X_t$ extends to a linear map $I : \mathscr{I}_T \to \mathscr{H}_T$ with the property that $I(1_{(0,t]}) = X_t$ and for $f, g \in \mathscr{I}_T$,

$$\mathbb{E}I(f)\overline{I(g)} = \langle \hat{f}, \hat{g} \rangle_{\mu_H}.$$

We denote the random variable $I(f)$ by $\int f \, \mathrm{d}X$ or $\int_0^T f(t) \, \mathrm{d}X_t$, and call it the Wiener integral of f with respect to X. (We note that in general not every element of $\mathscr{H}_T$ can be represented as such an integral, since for $H > \frac{1}{2}$ the space $\mathscr{I}_T$ is not complete, see [24].)

5.1. Fundamental martingale

For every $t \geqslant 0$, define the random variable $M_t \in \mathscr{H}_t$ as the image under the spectral isometry of the function $S_t(0, \cdot) \in \mathscr{L}_t$. Then in particular, M is adapted to the filtration generated by the fBm X. From the reproducing property of S_t it immediately follows that M has uncorrelated increments, i.e. it is a martingale. Indeed, for $s \leqslant t \leqslant u$ we have

$$\begin{aligned} \mathbb{E}X_s(M_u - M_t) &= \langle \hat{1}_{(0,s]}, S_u(0, \cdot) \rangle_{\mu_H} - \langle \hat{1}_{(0,s]}, S_t(0, \cdot) \rangle_{\mu_H} \\ &= \hat{1}_{(0,s]}(0) - \hat{1}_{(0,s]}(0) = 0. \end{aligned}$$

For the variance function of M we have

$$\mathbb{E}M_t^2 = \|S_t(0, \cdot)\|^2_{L^2(\mu_H)} = S_t(0, 0) = V_t.$$

We will see below that M is the martingale considered for instance by Norros et al. [21]. In accordance with their terminology, we call M the *fundamental martingale*.

We wish to show that M_t is in fact a Wiener integral of some deterministic kernel $k_t \in \mathscr{I}_t$ with respect to the fBm X. In spectral terms, this means we have to show that $S_t(0, \lambda)$ is the Fourier transform of some square integrable kernel. By substituting $\omega = 0$ in the formula given in Theorem 4.4 and using the basic property

$$z^{-\nu} J_\nu(z) \to \frac{1}{2^\nu \Gamma(\nu + 1)} \quad \text{as } z \to 0 \tag{5.1}$$

(cf. [16, Formula (5.16.1)]) we see that

$$S_t(0,\lambda)=\frac{\sqrt{\pi}}{2H\Gamma(H+1/2)}\left(\frac{t}{\lambda}\right)^{1-H}\mathrm{e}^{\mathrm{i}\frac{\lambda t}{2}}J_{1-H}\left(\frac{\lambda t}{2}\right).$$

To write this as a Fourier transform we use Poisson's integral representation of the Bessel function, which reads

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\int_{-1}^{1}\cos(zx)(1-x^2)^{\nu-1/2}\,\mathrm{d}x,\quad \nu>-\frac{1}{2} \tag{5.2}$$

(see e.g. [16, Formula (5.10.2)]). Rather straightforward computations now show that

$$S_t(0,\lambda)=\hat{k}_t(\lambda),$$

where

$$k_t(u)=\frac{u^{1/2-H}(t-u)^{1/2-H}}{2H\Gamma(1/2+H)\Gamma(3/2-H)},\quad u\leqslant t. \tag{5.3}$$

See for instance the proof of Dzhaparidze and Ferreira [9, Proposition 2.2], where the computation is carried out in the reverse direction.

Hence, we have arrived at the following well-known theorem (cf. e.g. [20,21,23]).

Theorem 5.1. *For* $t\geqslant 0$, *let the kernel* k_t *be defined by* (5.3). *We have that* $k_t\in\mathscr{I}_t$, *and the process* M *defined by*

$$M_t=\int_0^t k_t(u)\,\mathrm{d}X_u,\quad t\geqslant 0,$$

is a martingale with variance function $\mathbb{E}M_t^2=V_t$.

Let us remark (as in [11]) that the Poisson formula also shows that $S_t(0,\cdot)$ can be expressed in terms of fractional integrals, namely

$$S_t(0,\lambda)=\frac{1}{\Gamma(1/2+H)}I_{0+}^{3/2-H}(u^{1/2-H}\mathrm{e}^{\mathrm{i}\lambda u})(t). \tag{5.4}$$

This establishes the well-known connection between the fBm and the fractional calculus. (See [25] for background on fractional calculus.)

5.2. Moving average representation

It follows from Corollary 3.2 that $\mathscr{H}_t=\overline{\mathrm{sp}}\{M_u: u\leqslant t\}$. In particular, this means that X_t is a linear functional of $(M_u)_{u\leqslant t}$. The aim is now to write

$$X_t=\int_0^t l_t(u)\,\mathrm{d}M_u \tag{5.5}$$

for some explicit kernel $l_t\in L^2([0,t],V)$. This moving average representation of the fBm is the converse of Theorem 5.1.

In spectral terms, we have to determine l_t such that

$$\hat{1}_{(0,t]}(\lambda) = \int_0^t l_t(u)\,\mathrm{d}S_u(0,\lambda).$$

By (3.6) we have $\mathrm{d}S_u(0,\lambda) = \varphi(u,\lambda)\,\mathrm{d}V(u)$, so that the latter equation is equivalent with $\hat{1}_{(0,t]} = \mathscr{U} l_t$. It follows that (5.5) holds with $l_t = \mathscr{U}^{-1}\hat{1}_{(0,t]}$.

An explicit expression for l_t is most easily obtained by using some fractional calculus. Indeed, it follows from (3.6), (5.4) and fractional integration by parts that

$$\Gamma(1/2+H)\mathscr{U}f(\lambda) = \int_0^T \mathrm{e}^{\mathrm{i}\lambda t} t^{1/2-H}(I_{T-}^{1/2-H}f)(t)\,\mathrm{d}t, \tag{5.6}$$

for each $f \in L^2([0,T],V)$ for which the fractional integral is well defined. Taking $f = \Gamma(1/2+H)I_{T-}^{H-1/2}(u^{H-1/2}1_{(0,T]}(u))$ we obtain $\mathscr{U}f(\lambda) = \hat{1}_{(0,T]}(\lambda)$. Hence, we have proved the following theorem (cf. e.g. [20,6,21,23,24]).

Theorem 5.2. *For $t \geqslant 0$, let the kernel l_t be defined by*

$$l_t(u) = \Gamma(1/2+H)I_{t-}^{H-1/2}(v^{H-1/2}1_{(0,t]}(v))(u), \quad u \leqslant t.$$

We have that $l_t \in L^2([0,t],V)$ and (5.5) holds for all $t \geqslant 0$.

Evaluation of the fractional integral yields the more explicit expression

$$l_t(u) = t^{H-1/2}(t-u)^{H-1/2} - \int_u^t (t-v)^{H-1/2}\,\mathrm{d}v^{H-1/2}, \quad u \leqslant t.$$

Since M is a continuous Gaussian martingale with bracket $\langle M\rangle_t = V_t$, we have $\mathrm{d}M_u = \sqrt{V'_u}\,\mathrm{d}W_u$, where W is a standard Brownian motion. Hence, the moving average representation can be rephrased as follows.

Corollary 5.3. *For $t \geqslant 0$, let the kernel w_t be defined by $w_t(u) = l_t(u)\sqrt{V'_u}$ for $u \leqslant t$. Then (1.1) holds for $t \geqslant 0$, with W a standard Brownian motion.*

Also observe that the computations above can easily be generalized to more general integration kernels. For $r_t \in \mathscr{I}_t$ it holds that

$$\begin{aligned}\int_0^t r_t(u)\,\mathrm{d}X_u &= \int_0^t \mathscr{U}^{-1}\hat{r}_t(u)\,\mathrm{d}M_u \\ &= \Gamma(1/2+H)\int_0^t I_{t-}^{H-1/2}(v^{H-1/2}r_t(v))(u)\,\mathrm{d}M_u,\end{aligned}$$

provided the fractional integral is well defined. Conversely, (5.6) shows that if $f \in L^2([0,t],V)$ and $I_{t-}^{1/2-H}f$ exists and also belongs to $L^2([0,t],V)$, then

$$\Gamma(1/2+H)\int_0^t f(u)\,\mathrm{d}M_u = \int_0^t u^{1/2-H}(I_{t-}^{1/2-H}f)(u)\,\mathrm{d}X_u.$$

5.3. Series representation

We can argue exactly as in Dzhaparidze and Van Zanten [11] to deduce the following series representation of the fBm from Theorem 4.6.

Theorem 5.4. *Let* $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ *be the real-valued zeros of the Bessel function* J_{1-H}. *We have*

$$X_t = \sum_{n\in\mathbb{Z}} \hat{1}_{(0,t]}\left(\frac{2\omega_n}{T}\right) Z_n, \quad t \leqslant T,$$

where the Z_n *are independent, centered Gaussian random variables with variance*

$$\mathbb{E}|Z_n|^2 = \frac{1}{S_T(2\omega_n/T, 2\omega_n/T)}.$$

With probability one, the series converges uniformly in $t \in [0, T]$.

Proof. By Theorem 4.6 we have, for $t \leqslant T$,

$$\hat{1}_{(0,t]}(\lambda) = \sum_{n\in\mathbb{Z}} \hat{1}_{(0,t]}\left(\frac{2\omega_n}{T}\right) \frac{S_T(2\omega_n/T, \lambda)}{S_T(2\omega_n/T, 2\omega_n/T)}$$

in $\mathscr{L}_T$. It follows that for Z_n the random variable corresponding to the function $S_T(2\omega_n/T, \cdot)/S_T(2\omega_n/T, 2\omega_n/T)$ under the spectral isometry, the series representation holds in L^2. By the reproducing property, we have

$$\mathbb{E}|Z_n|^2 = \frac{1}{S_T^2(2\omega_n/T, 2\omega_n/T)} \int |S_T(2\omega_n/T, \lambda)|^2 \, \mu_H(\mathrm{d}\lambda) = \frac{1}{S_T(2\omega_n/T, 2\omega_n/T)}.$$

The almost sure uniform convergence can be proved using the Itô–Nisio theorem. See e.g. the proof of Dzhaparidze and Van Zanten [10, Theorem 4.5]. □

The proof of Theorem 5.4 shows that we can just as easily obtain series expansions of Wiener integrals with respect to the fBm. Indeed, if $f \in \mathscr{I}_T$ it holds that

$$\int_0^T f(u)\,\mathrm{d}X_u = \sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{2\omega_n}{T}\right) Z_n,$$

with the same Z_n as in the statement of the theorem.

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